

# Decomposing graphs into a constant number of locally irregular subgraphs

Julien Bensmail\*

Martin Merker\*

Carsten Thomassen\*

Department of Applied Mathematics and Computer Science  
Technical University of Denmark  
DK-2800 Lyngby, Denmark

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## Abstract

A graph is *locally irregular* if no two adjacent vertices have the same degree. The *irregular chromatic index*  $\chi'_{\text{irr}}(G)$  of a graph  $G$  is the smallest number of locally irregular subgraphs needed to edge-decompose  $G$ . Not all graphs have such a decomposition, but Baudon, Bensmail, Przybyło, and Woźniak conjectured that if  $G$  can be decomposed into locally irregular subgraphs, then  $\chi'_{\text{irr}}(G) \leq 3$ . In support of this conjecture, Przybyło showed that  $\chi'_{\text{irr}}(G) \leq 3$  holds whenever  $G$  has minimum degree at least  $10^{10}$ .

Here we prove that every bipartite graph  $G$  which is not an odd length path satisfies  $\chi'_{\text{irr}}(G) \leq 10$ . This is the first general constant upper bound on the irregular chromatic index of bipartite graphs. Combining this result with Przybyło's result, we show that  $\chi'_{\text{irr}}(G) \leq 328$  for every graph  $G$  which admits a decomposition into locally irregular subgraphs. Finally, we show that  $\chi'_{\text{irr}}(G) \leq 2$  for every 16-edge-connected bipartite graph  $G$ .

## 1 Introduction

A graph  $G$  is *locally irregular* if any two adjacent vertices have distinct degrees. This concept of local irregularity was investigated by Karoński, Łuczak, and Thomason in [5], where they introduced the so-called 1-2-3 Conjecture. An edge-weighting of  $G$  is called *neighbour-sum-distinguishing*, if for every two adjacent vertices of  $G$  the sums of incident weights are distinct. The least number  $k$  for which  $G$  admits a neighbour-sum-distinguishing edge-weighting using weights  $1, 2, \dots, k$  is denoted  $\chi'_{\Sigma}(G)$ .

**Conjecture 1.1** (1-2-3 Conjecture [5]). *For every graph  $G$  with no component isomorphic to  $K_2$ , we have  $\chi'_{\Sigma}(G) \leq 3$ .*

This conjecture is equivalent to stating that a graph can be made locally irregular by replacing some of its edges by two or three parallel edges. Although the 1-2-3 Conjecture has received considerable attention in the last decade, it is still an open question. The best result so far was shown by Kalkowski, Karoński, and Pfender [4] who proved  $\chi'_{\Sigma}(G) \leq 5$

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whenever  $G$  has no component isomorphic to  $K_2$ . For more details, we refer the reader to the survey by Seamone [8] on the 1-2-3 Conjecture and related problems.

A different approach was taken by Baudon, Bensmail, Przybyło, and Woźniak in [1], where edge-decompositions of graphs into locally irregular graphs were studied. From now on, all graphs we consider are simple and finite. A decomposition into locally irregular subgraphs can be regarded as an improper edge-colouring where each colour class induces a locally irregular graph. We call such an edge-colouring *locally irregular*. If  $G$  admits a locally irregular edge-colouring, then we call  $G$  *decomposable*. For every decomposable graph  $G$ , we define the *irregular chromatic index* of  $G$ , denoted by  $\chi'_{\text{irr}}(G)$ , as the least number of colours in a locally irregular edge-colouring of  $G$ . If  $G$  is not decomposable, then  $\chi'_{\text{irr}}(G)$  is not defined and we call  $G$  *exceptional*. The following conjecture has a similar flavour to the 1-2-3 Conjecture.

**Conjecture 1.2** ([1]). *For every decomposable graph  $G$ , we have  $\chi'_{\text{irr}}(G) \leq 3$ .*

Every connected graph of even size can be decomposed into paths of length 2 and is thus decomposable. Hence, all exceptional graphs have odd size and a complete characterisation of exceptional graphs was given in [1]. To state this characterisation, we first need to define a family  $\mathcal{T}$  of graphs. A connected graph  $G$  belongs to  $\mathcal{T}$  if and only if  $G$  has a nonempty collection of triangles,  $G$  has no other cycles,  $G$  has maximum degree at most 3, all vertices not in a triangle have degree at most 2, and if  $P$  is path in  $G$  whose intermediate vertices all have degree 2 in  $G$  and  $P$  is maximal with this property, then  $P$  has odd length if and only if both ends are in triangles.

**Theorem 1.3** ([1]). *A connected graph is exceptional, if and only if it is (1) a path of odd length, (2) a cycle of odd length, or (3) a member of  $\mathcal{T}$ .*

The number 3 in Conjecture 1.2 cannot be decreased to 2, as shown for example by cycles with length congruent to 2 modulo 4 and complete graphs. In [1], Conjecture 1.2 was verified for several classes of graphs such as trees, complete graphs, and regular graphs with degree at least  $10^7$ . In [2], Baudon, Bensmail, and Sopena showed that determining the irregular chromatic index of a graph is NP-complete in general, and that, although infinitely many trees have irregular chromatic index 3, the same problem for trees can be solved in linear time. More recently, Przybyło [7] gave further evidence for Conjecture 1.2 by verifying it for graphs of large minimum degree.

**Theorem 1.4** ([7]). *For every graph  $G$  with minimum degree at least  $10^{10}$ , we have  $\chi'_{\text{irr}}(G) \leq 3$ .*

Despite this result, Conjecture 1.2 is still wide open, even in much weaker forms. Until now it was not known whether there exists a constant  $c$  such that  $\chi'_{\text{irr}}(G) \leq c$  holds for every decomposable graph  $G$ . This was also an open problem when restricted to bipartite graphs, see [1, 2, 3, 7].

In this paper we show  $\chi'_{\text{irr}}(G) \leq 328$  for every decomposable graph  $G$ , hence providing the first constant upper bound on the irregular chromatic index. The proof consists of three steps. First, we show in Section 2 that we can restrict our attention to connected graphs of even size. Notice that every connected graph of even size can be decomposed into paths of length 2 and is thus decomposable. We show that every decomposable graph  $G$  of odd size contains a locally irregular subgraph  $H$  such that all connected components of  $G - E(H)$  have even size. In Section 3, we investigate bipartite graphs  $G$  of even size and show that

$\chi'_{\text{irr}}(G) \leq 9$  holds in this case. Finally, in Section 4 we decompose a connected graph  $G$  of even size into a graph  $H$  of minimum degree  $10^{10}$  and a  $(2 \cdot 10^{10})$ -degenerate graph  $D$  in which every component has even size. We use Theorem 1.4 to decompose  $H$ , and we further decompose  $D$  into 36 bipartite graphs of even size. By using our result for bipartite graphs, this results in a decomposition of  $G$  into  $3 + 9 \cdot 36 = 327$  locally irregular subgraphs.

We have proved a constant upper bound on the irregular chromatic index, but there is still a significant gap between our result and the conjectured value 3. We conclude this article by showing in Section 5 that  $\chi'_{\text{irr}}(G) \leq 2$  for every 16-edge-connected bipartite graph  $G$ .

## 2 Reduction to graphs of even size

In this section we show that the weakening of Conjecture 1.2, where 3 is replaced by a larger number, can be reduced to connected graphs of even size. More precisely, given a decomposable graph  $G$  with odd size, we can always remove a locally irregular subgraph  $H$  from  $G$ , so that all connected components of  $G - E(H)$  have even size.

**Lemma 2.1.** *Let  $G$  be a connected graph of odd size. For every vertex  $v \in V(G)$  there exists an edge  $e$  incident with  $v$  such that every connected component of  $G - e$  has even size.*

*Proof.* Let  $E(v)$  denote the set of edges incident with  $v$ . If  $e \in E(v)$  is not a cut-edge, then  $G - e$  is connected and of even size. We may thus assume that all edges in  $E(v)$  are cut-edges. For every  $e \in E(v)$ , let  $H_e$  denote the connected component of  $G - e$  not containing  $v$ . Now

$$E(G) = \bigcup_{e \in E(v)} E(H_e) \cup \{e\}.$$

Since  $|E(G)|$  is odd, there exists  $e \in E(v)$  for which  $|E(H_e) \cup \{e\}|$  is odd. Thus,  $H_e$  is of even size, and so is the other connected component of  $G - e$ .  $\square$

**Lemma 2.2.** *Let  $G$  be a connected graph of even size. For every vertex  $v \in V(G)$  there exists a path  $P$  of length 2 containing  $v$  such that every connected component of  $G - E(P)$  has even size.*

*Proof.* Let  $e$  be an edge incident with  $v$ . Then  $G - e$  has precisely one connected component of odd size, and  $e$  is incident with a vertex  $u$  of that component, possibly  $u = v$ . By Lemma 2.1 we can delete an edge  $f$  incident with  $u$  so that every component of  $G - \{e, f\}$  has even size. Since  $e$  and  $f$  are incident, they form a path  $P$  of length 2.  $\square$

**Theorem 2.3.** *Let  $G$  be a connected graph of odd size. If  $G$  is decomposable, then  $G$  contains a locally irregular subgraph  $H$  such that every connected component of  $G - E(H)$  has even size.*

*Proof.* We show that we can choose  $H$  to be isomorphic to  $K_{1,3}$  or to  $K_{1,3}$  where two edges are subdivided once. Assume that  $G$  is a graph for which we cannot delete one of these two graphs such that every connected component in the resulting graph is of even size. If  $G$  has maximum degree at most 2 and odd size, then  $G$  is exceptional. We can thus assume that  $G$  has maximum degree at least 3. Notice that every vertex  $v$  of degree at least 3 in  $G$  must be a cutvertex, since otherwise we can delete a claw (i.e. a subgraph isomorphic to  $K_{1,3}$ ) centred at  $v$ .

First, suppose that  $G$  contains a cycle  $C$ . Let  $V_C$  denote the vertices of  $C$  with degree at least 3. For every  $v \in V_C$ , let  $E_C(v)$  denote the two edges of  $C$  that are incident with  $v$ . If  $G - E_C(v)$  is connected, then we can use Lemma 2.1 to delete one more edge at  $v$  so that every connected component in the resulting graph has even size. We may thus assume that  $G - E_C(v)$  is disconnected. Let  $G_C(v)$  denote the connected component of  $G - E_C(v)$  containing  $v$ . If  $|E(G_C(v))|$  is odd, then we can again use Lemma 2.1 to delete one more edge at  $v$  to reach the desired conclusion. Thus we may assume that  $|E(G_C(v))|$  is even for all  $v \in V_C$ . By Lemma 2.2, there exists a path  $P_v$  of length 2 in  $G_C(v)$  incident with  $v$  such that every connected component of  $G_C(v) - E(P_v)$  has even size. If  $v$  is the middle vertex of  $P_v$ , then  $P_v$  together with one of the two edges in  $E_C(v)$  forms a claw whose removal leaves a graph where every connected component has even size. Thus, we may assume that  $v$  is an endvertex of  $P_v$ . If  $C$  has length at least 4, then let  $P_C$  be a path of length 3 in  $C$  in which  $v$  has degree 2. The graph  $P_v \cup P_C$  is locally irregular and it is easy to see that every connected component of  $G - E(P_v) - E(P_C)$  has even size.

Thus we may assume that all cycles of  $G$  have length 3. Suppose two cycles  $C_1, C_2$  have a vertex  $v$  in common. Choose an edge  $e_i$  incident with  $v$  in  $C_i$  for  $i \in \{1, 2\}$ . Now  $G - \{e_1, e_2\}$  is connected, so we can apply Lemma 2.1 to delete one more edge at  $v$  so that every connected component has even size.

So far, we have shown that triangles are the only cycles in  $G$  and that any two triangles are disjoint. Now we show that there exists no induced claw in  $G$ . Suppose for a contradiction that  $v$  is a vertex of degree at least 3 which is a center of a claw. If  $v$  is contained in a triangle, then we assume that the degree of  $v$  is at least 4. Since any two triangles are disjoint, there exists at most one edge between the neighbours of  $v$ . By Lemma 2.1, we can delete an edge  $uv$  so that every component of  $G' = G - uv$  has even size. By our choice of  $v$ , there exists two neighbours  $u_1$  and  $u_2$  of  $v$  such that  $\{u, u_1, u_2\}$  is an independent set in  $G$ . Let  $G_1$  denote the connected component of odd size in  $G' - u_1v$ . If  $G_1$  contains  $v$ , then we can delete a third edge  $e$  at  $v$  by Lemma 2.1 such that all components of  $G' - u_1v - e$  have even size. Thus, we can assume that  $G_1$  contains  $u_1$  but not  $v$ . Similarly, we may assume that the odd component  $G_2$  of  $G' - u_2v$  contains  $u_2$  but not  $v$ . Now we can apply Lemma 2.1 to delete an edge  $e_i$  incident with  $u_i$  in  $G_i$  such that every connected component of  $G_i - e_i$  has even size for  $i \in \{1, 2\}$ . Thus, every connected component of  $G' - e_1 - e_2 - u_1v - u_2v$  has even size. Since  $G_1, G_2$  are distinct components of  $G - v$ , the graph we removed is isomorphic to  $K_{1,3}$  where two edges are subdivided once. This contradicts our choice of  $G$ , implying that  $G$  has no induced claw.

Thus we may assume that the maximum degree in  $G$  is 3 and that every vertex of degree 3 is contained in a triangle. Since there are no other cycles, this implies that the contraction of all triangles results in a tree of maximum degree 3. All that remains to show is that the parities of the path lengths are the same as for the exceptional graphs. Let  $P$  be a path joining a leaf in  $G$  with a triangle  $C$ . Let  $v$  be the common vertex of  $P$  and  $C$ . Now  $P = G_C(v)$  and since  $|E(G_C(v))|$  is even, the length of  $P$  is even. Finally, let  $P$  be a path joining two different triangles  $C_1$  and  $C_2$ . If  $v_1$  and  $v_2$  denote the endvertices, then

$$|E(G)| = |E(G_{C_1}(v_1))| + |E(G_{C_2}(v_2))| - |E(P)|.$$

Since  $|E(G_{C_1}(v_1))|$  and  $|E(G_{C_2}(v_2))|$  are even and  $|E(G)|$  is odd, we get that  $|E(P)|$  must also be odd. This shows that  $G$  is exceptional.  $\square$

### 3 Locally irregular decompositions of bipartite graphs

We now focus on the irregular chromatic index of bipartite graphs. Recall that the only bipartite exceptional graphs are odd length paths. In Corollary 3.10 we show  $\chi'_{\text{irr}}(G) \leq 10$  for every decomposable bipartite graph  $G$ , which is the first constant upper bound on  $\chi'_{\text{irr}}$  for bipartite graphs.

If all vertices in one partition class of the bipartite graph  $G$  have even degree, while the vertices in the other partition class have odd degree, then  $G$  is locally irregular. The idea of the proof is to remove some well-behaved subgraphs from  $G$  to obtain a graph which is very close to this structure. These well-behaved subgraphs include a particular kind of forest, which is defined as follows.

**Definition 3.1.** *We say a forest is **balanced** if it has a bipartition such that all vertices in one of the partition classes have even degree.*

Since a balanced forest cannot contain an odd length path as a connected component, it follows from [1] that  $\chi'_{\text{irr}}(F) \leq 3$  for every balanced forest  $F$ . The characterisation of trees  $T$  with  $\chi'_{\text{irr}}(T) \leq 2$  in [2] implies that even  $\chi'_{\text{irr}}(F) \leq 2$  holds for balanced forests  $F$ . For the sake of completeness, we present a short proof of this special case.

**Lemma 3.2.** *If  $F$  is a balanced forest, then  $F$  has a 2-colouring of the edges, such that each colour induces a locally irregular graph, and, for each vertex  $v$  in the partition class with no odd degree vertex, all edges incident with  $v$  have the same colour. In particular,  $\chi'_{\text{irr}}(F) \leq 2$ .*

*Proof.* The proof is by induction on the number of edges of  $F$ . Clearly, we may assume that  $F$  is connected. Let  $A$  and  $B$  be the partition classes of  $F$ , where all vertices in  $B$  have even degree. We may assume that some vertex in  $A$  has even degree since otherwise we can give all edges of  $F$  the same colour. Let  $v$  be a vertex in  $A$  of even degree  $q$ . We delete  $v$  but keep the edges incident with  $v$  and let them go to  $q$  new vertices  $v_1, v_2, \dots, v_q$  each of degree 1. In other words, we split  $F$  into  $q$  new trees  $T_1, T_2, \dots, T_q$  such that the union of their edges is the edge set of  $F$ . Each of the trees  $T_1, T_2, \dots, T_q$  is balanced and has therefore a colouring of its edges in colours red and blue satisfying the conclusion of Lemma 3.2. This also gives a colouring of the edges of  $F$  in colours red and blue. By switchings colours in some of the  $T_i$ , if necessary, we can ensure that the red degree of  $v$  is 1. This shows that also  $F$  satisfies the conclusion of Lemma 3.2.  $\square$

Apart from balanced forests we shall also delete a subgraph which is the union of a path and an induced cycle. The following lemma gives an upper bound on the irregular chromatic index in this case.

**Lemma 3.3.** *Let  $G$  be a bipartite graph and let  $v$  be a vertex in  $G$ . If  $G$  is the edge-disjoint union of an induced cycle  $C$  through  $v$  and a path  $P$  starting at  $v$ , then  $\chi'_{\text{irr}}(G) \leq 4$ .*

*Proof.* If the length of  $P$  is 0, then  $\chi'_{\text{irr}}(G) \leq 3$ , so we may assume  $P$  has positive length. First suppose that  $P$  has odd length. Let  $e$  denote the edge of  $P$  incident with  $v$ . It is easy to see that  $\chi'_{\text{irr}}(C + e) \leq 2$ . Thus,

$$\chi'_{\text{irr}}(G) \leq \chi'_{\text{irr}}(C + e) + \chi'_{\text{irr}}(P - e) \leq 2 + 2 = 4.$$

Now suppose the length of  $P$  is even. If the length of  $C$  is divisible by 4, then

$$\chi'_{\text{irr}}(G) \leq \chi'_{\text{irr}}(C) + \chi'_{\text{irr}}(P) \leq 2 + 2 = 4.$$

We may therefore assume the length of  $C$  is congruent to 2 modulo 4. Let  $e$  denote the edge of  $P$  incident with  $v$ , and let  $f$  denote the edge incident with  $e$  on  $P$ . It is easy to check that if  $e$ ,  $f$  and all edges of  $C$  incident to  $e$  or  $f$  are coloured 1, then this colouring can be extended to a locally irregular  $\{1, 2\}$ -edge-colouring of  $C + e + f$ . Thus, we have

$$\chi'_{\text{irr}}(G) \leq \chi'_{\text{irr}}(C + e + f) + \chi'_{\text{irr}}(P - e - f) \leq 2 + 2 = 4.$$

□

The following lemma is well-known.

**Lemma 3.4.** *Let  $G$  be a connected graph and let  $S$  be a set of vertices. If  $S$  is even, then there exists a collection of  $\frac{|S|}{2}$  edge-disjoint paths in  $G$  such that each vertex in  $S$  is an endvertex of precisely one of them.*

*Proof.* Take a collection of paths having the vertices in  $S$  as endvertices for which the total length is minimal. □

**Corollary 3.5.** *If  $G$  is a connected bipartite graph of even size with partition classes  $A$  and  $B$ , then there exists a balanced forest  $F$  with leaves in  $A$  such that in  $G - E(F)$  all vertices in  $A$  have even degree.*

*Proof.* Notice that since  $G$  has even size, the number of vertices in  $A$  with odd degree is even. The statement follows by choosing  $S$  to be the set of odd-degree vertices in  $A$ , and  $F$  as the union of the paths given by Lemma 3.4. □

**Corollary 3.6.** *Let  $G$  be a connected bipartite graph with partition classes  $A$  and  $B$ , and let  $v$  be a vertex in  $B$ . If all vertices in  $A$  have even degree, then there exists a balanced forest  $F$  with leaves in  $B$  such that in  $G - E(F)$  all vertices in  $B \setminus \{v\}$  have odd degree.*

*Proof.* Choose  $S$  as the set of even-degree vertices in  $B$ . If  $|S|$  is odd, then we apply Lemma 3.4 to the set  $S \cup \{v\}$  or  $S \setminus \{v\}$ , and if  $|S|$  is even we apply Lemma 3.4 to the set  $S$ . The union of the paths is the desired balanced forest. □

**Lemma 3.7.** *Let  $G$  be a bipartite graph with partition classes  $A$  and  $B$ , and let  $v$  be a vertex in  $B$ . If all vertices in  $A$  have even degree and all vertices in  $B \setminus \{v\}$  have odd degree, then there exists a path  $P$  starting in  $v$  such that  $G - E(P)$  is locally irregular.*

*Proof.* If  $v$  has odd degree, then we can choose  $P$  as a path of length 0. If  $v$  has even degree and  $G$  is not locally irregular, then  $v$  is adjacent to a vertex  $u_1$  of the same degree. We choose the edge  $vu_1$  as the first edge of  $P$  and define  $G_1 = G - vu_1$ . If  $G_1$  is not locally irregular, then  $u_1$  is adjacent to a neighbour  $u_2$  of the same degree. In this case we extend  $P$  by the edge  $u_1u_2$  and define  $G_2 = G_1 - u_1u_2$ . We continue like this, defining  $G_{i+1}$  if  $G_i$  is not locally irregular by deleting a conflict edge  $u_iu_{i+1}$ . We claim that this process stops with a locally irregular graph  $G_k$  and that the deleted edges form a path. Notice that if  $G_i$  is not locally irregular, then  $u_i$  is incident to a vertex  $u_{i+1}$  of the same degree. Moreover, the degree of  $u_i$  in  $G_i$  is  $d(v) - i$ , so the degrees  $d(u_i)$  form a decreasing sequence. In particular,  $u_i \neq u_j$  for  $i \neq j$  and  $u_i \neq v$  for all  $i$ . Thus, eventually the process stops with a locally irregular graph  $G_k$  and  $G - E(G_k)$  is a path of length  $k$ . □

**Lemma 3.8.** *Let  $G$  be a bipartite graph with partition classes  $A$  and  $B$ . If all vertices in  $A$  have even degree, then  $\chi'_{\text{irr}}(G) \leq 7$ .*

*Proof.* We may assume that  $G$  is connected. By Lemma 3.6, we can delete a balanced forest  $F$  with leaves in  $B$  such that in the resulting graph  $G'$  there is at most one vertex of even degree in  $B$ , say  $v$ . If  $v$  does not exist or if  $v$  is an isolated vertex in  $G'$ , then  $G'$  is locally irregular and  $\chi'_{\text{irr}}(G) \leq \chi'_{\text{irr}}(F) + \chi'_{\text{irr}}(G') \leq 3$ . Thus, we may assume that  $v$  exists. Notice that  $G'$  might consist of several connected components, but every component not containing  $v$  is locally irregular. Let  $H$  denote the connected component of  $G'$  containing  $v$ .

If there exists no cycle through  $v$  in  $H$ , then all edges incident with  $v$  are cut-edges. Let  $e$  be an edge incident with  $v$ , and let  $H_1$  and  $H_2$  denote the two connected components of  $H - e$ . We may assume that  $H_1$  contains  $v$ . Notice that the degree of  $v$  in  $H_1$  and in  $H_2 + e$  is odd, while the degrees of its neighbours are even. It follows that both  $H_1$  and  $H_2 + e$  are locally irregular and hence

$$\chi'_{\text{irr}}(G) \leq \chi'_{\text{irr}}(F) + \chi'_{\text{irr}}(H_1) + \chi'_{\text{irr}}(H_2 + e) \leq 4.$$

Thus, we may assume that there exists a cycle going through  $v$ . Let  $C$  be a cycle through  $v$  of shortest length and set  $H' = H - E(C)$ . Since the parities of the degrees remain unchanged, the vertex  $v$  is still the only vertex in  $B$  that could have positive even degree in  $H'$ , while all vertices in  $A$  have even degree. By Lemma 3.7, there exists a path  $P$  in  $H'$  starting in  $v$  such that  $H' - E(P)$  is locally irregular. Now  $\chi'_{\text{irr}}(C \cup P) \leq 4$  by Lemma 3.3 and we have

$$\chi'_{\text{irr}}(G) \leq \chi'_{\text{irr}}(F) + \chi'_{\text{irr}}(H' - E(P)) + \chi'_{\text{irr}}(C \cup P) \leq 2 + 1 + 4 = 7.$$

□

We are now ready for the main result of this section.

**Theorem 3.9.** *If  $G$  is a connected bipartite graph of even size, then  $\chi'_{\text{irr}}(G) \leq 9$ .*

*Proof.* By Lemma 3.5, we can delete a balanced forest  $F$  of  $G$  so that the degrees in  $A$  in the resulting graph  $G'$  are even. By Lemma 3.2 we have  $\chi'_{\text{irr}}(F) \leq 2$ , and  $\chi'_{\text{irr}}(G') \leq 7$  follows from Lemma 3.8. Thus  $\chi'_{\text{irr}}(G) \leq \chi'_{\text{irr}}(F) + \chi'_{\text{irr}}(G') \leq 2 + 7 = 9$ . □

**Corollary 3.10.** *If  $G$  is a connected bipartite graph and not an odd length path, then  $\chi'_{\text{irr}}(G) \leq 10$ .*

*Proof.* Since paths of odd lengths are the only exceptional bipartite graphs, this follows immediately from Theorems 2.3 and 3.9. □

## 4 Locally irregular decompositions of degenerate graphs

Here we apply the result from the previous section by decomposing degenerate graphs into bipartite graphs of even size. We show that every connected  $d$ -degenerate graph of even size can be decomposed into at most  $\lceil \log_2(d+1) \rceil + 1$  bipartite graphs whose components all have even size. The proof makes repeated use of the following easy lemma.

**Lemma 4.1.** *If  $G$  is a graph with a vertex  $v$  such that  $G - v$  is bipartite, then there exists a set  $E$  of at most  $\lfloor \frac{d(v)}{2} \rfloor$  edges incident with  $v$  such that  $G - E$  is bipartite.*

*Proof.* Since  $G - v$  is bipartite, there exists a partition class containing at most  $\lfloor \frac{d(v)}{2} \rfloor$  neighbours of  $v$ . Deleting all edges in  $G$  from  $v$  to these vertices results in a bipartite graph. □

**Lemma 4.2.** *Let  $d$  be an even natural number,  $\ell \geq \lceil \log_2 d \rceil + 1$ , and  $v$  a vertex of degree  $d$  in a graph  $G$ . If  $G - v$  is the edge-disjoint union of  $\ell$  bipartite graphs in which every component has even size, then so is  $G$ .*

*Proof.* Notice that it suffices to prove the statement for  $\ell = \lceil \log_2 d \rceil + 1$ . We use induction on  $d$ . In the case  $d = 2$  we colour  $G - v$  with colours 1 and 2 so that the monochromatic connected components are bipartite subgraphs of even size. Let  $u_1, u_2$  be the neighbours of  $v$  in  $G$ . If  $u_1$  and  $u_2$  are not connected by an odd length path in colour 1, then colouring both  $vu_1$  and  $vu_2$  with colour 1 will keep all monochromatic components bipartite and of even size. Thus, we may assume that  $u_1$  and  $u_2$  are connected by a monochromatic path of odd length in each colour. Let  $P = v_0v_1 \dots v_k$  be a monochromatic path from  $u_1$  to  $u_2$  in colour 2, so  $v_0 = u_1$  and  $v_k = u_2$ . Suppose that for every  $i \in \{0, \dots, k-1\}$  there exists an even length path in colour 1 from  $v_i$  to  $v_{i+1}$ . By concatenating them, we get a walk of even length from  $v_0$  to  $v_k$ . Since there is also a path of odd length joining  $v_0$  and  $v_k$  in colour 1, this contradicts the assumption that the subgraph in colour 1 is bipartite. Thus, there exists  $i \in \{0, \dots, k-1\}$  for which there is no even length path in colour 1 from  $v_i$  to  $v_{i+1}$ . Choose  $i$  minimal with this property. We change the colour of  $v_iv_{i+1}$  to colour 1. By the choice of  $i$ , all monochromatic components in colour 1 are still bipartite. Now there exists precisely one monochromatic component of odd size in each colour. Notice that the monochromatic component of odd size in colour 1 is incident with both  $u_1$  and  $u_2$ , while the one in colour 2 is incident with at least one of  $u_1$  and  $u_2$ . Thus, we can colour one of the edges at  $v$  with colour 2 so that all monochromatic components in colour 2 are bipartite and of even size. Colouring the other edge at  $v$  with colour 1 yields the desired decomposition.

Now suppose  $d \geq 4$  and that the statement is true for all smaller even numbers. Set  $d' = \frac{d}{2}$  if  $d$  is divisible by 4, and  $d' = \frac{d}{2} + 1$  otherwise. Notice that  $d'$  is even and  $\lceil \log_2 d \rceil = \lceil \log_2 d' \rceil + 1$ . Let  $\mathcal{H}$  be the collection of  $\lceil \log_2 d \rceil + 1$  bipartite graphs in  $G - v$  with even component sizes. Choose  $H \in \mathcal{H}$  and denote by  $G_H$  the graph we get by adding  $v$  and all its incident edges to  $H$ . By Lemma 4.1, there exists a set  $E$  of  $d'$  edges incident with  $v$  such that  $G_H - E$  is bipartite. Since  $d - d'$  is even, all connected components of  $G_H - E$  have even size. We add the edges in  $E$  to the union of the graphs in  $\mathcal{H} \setminus \{H\}$  to obtain a graph  $G'$ . By the induction hypothesis, we can decompose  $G'$  into  $\lceil \log_2 d' \rceil + 1$  bipartite graphs where every component has even size. Together with  $G_H - E$ , this is a collection of  $\lceil \log_2 d' \rceil + 2 = \lceil \log_2 d \rceil + 1$  such graphs.  $\square$

Notice that the bound  $\lceil \log_2 d \rceil + 1$  can in general not be decreased by more than 1. The complete graph  $K_{d+1}$  is  $d$ -degenerate and at least  $\lceil \log_2(d+1) \rceil$  bipartite graphs are needed to decompose it. Moreover, we might need more bipartite graphs to achieve that all components have even size. For example, the complete graph  $K_4$  can be decomposed into two bipartite graphs, but three bipartite graphs are necessary to achieve even component sizes.

**Theorem 4.3.** *Let  $d \geq 1$  be a natural number. If  $G$  is a  $d$ -degenerate graph in which every connected component has even size, then  $G$  can be decomposed into  $\lceil \log_2(d+1) \rceil + 1$  bipartite graphs in which all connected components have even size.*

*Proof.* Suppose not, and let  $G$  be a smallest counterexample. Clearly  $G$  is connected.

**Claim.** *If  $v$  is a cutvertex of  $G$ , then  $v$  is adjacent to precisely one vertex  $u$  of degree 1 and  $G - u - v$  is connected.*



To prove the claim, suppose there exists a 1-separation  $\{V_1, V_2\}$  of  $G$  with  $V_1 \cap V_2 = \{v\}$  and  $|V_1|, |V_2| \geq 3$ . If  $G[V_1]$  and  $G[V_2]$  have even size, then we can decompose  $G[V_1]$  and  $G[V_2]$  by induction. If  $G[V_1]$  and  $G[V_2]$  have odd size, then we construct two new graphs  $H_1$  and  $H_2$  by adding a new vertex  $v_i$  to  $G[V_i]$  together with the single edge  $vv_i$ . Since  $|V_1|, |V_2| \geq 3$ , both  $H_1$  and  $H_2$  are smaller than  $G$  so we can decompose them by induction. We think of the decomposition as an edge-colouring, and we permute colours so that the edges  $vv_i$  receive the same colour in both subgraphs. This corresponds to a colouring of  $G$  in which every monochromatic component is bipartite and of even size. This proves the claim.

In particular, every vertex is adjacent to at most one vertex of degree 1. Among all vertices of degree greater than 1, let  $v$  be one of minimal degree. Since  $G$  is  $d$ -degenerate, we have  $d(v) \leq d+1$ . Suppose first that  $d(v)$  is even. Since  $G$  is a smallest counterexample, we can decompose  $G - v$  into  $\lceil \log_2(d+1) \rceil + 1$  bipartite graphs in which all connected components have even size. By Lemma 4.2, this gives rise to the desired decomposition of  $G$ .

We may thus assume that  $d(v)$  is odd. Set  $d' = \frac{1}{2}(d(v) - 1)$  if  $d(v)$  is congruent to 1 modulo 4, and  $d' = \frac{1}{2}(d(v) + 1)$  otherwise. Notice that  $d'$  is even and  $\lceil \log_2(d+1) \rceil \geq \lceil \log_2 d' \rceil + 1$ . Let  $u$  be a neighbour of  $v$  of degree greater than 1. If  $G - v$  has an isolated vertex, then we let  $w$  denote that vertex. Otherwise we add an isolated vertex  $w$ . The graph  $G - v + uw$  has even size, so we can decompose it as in the previous case. This gives us a decomposition of  $G - v$  into  $\lceil \log_2(d+1) \rceil + 1$  bipartite graphs in which all connected components are of even size, apart from one component of odd size which is incident with  $u$ . Let  $H$  be the bipartite subgraph of odd size, and let  $H_o$  be the connected component of odd size. Let  $G_H$  be the graph we get by adding  $v$  and all its incident edges to  $H$ . By Lemma 4.1, there exists a set  $E$  of precisely  $d'$  edges incident with  $v$  such that  $G_H - E$  is bipartite. We may assume that  $E$  does not contain all edges that are incident with  $H_o$ . Since  $d(v) - d'$  is odd, all connected components of  $G_H - E$  have even size. We add the edges in  $E$  to  $G - v - E(H)$  to obtain a graph  $G'$ . Notice that  $G - v - E(H)$  is the union of  $\lceil \log_2(d+1) \rceil$  bipartite graphs with components of even size. By Lemma 4.2, we can decompose  $G'$  into  $\lceil \log_2(d+1) \rceil$  bipartite graphs where every component has even size. Together with  $G_H - E$ , this is a collection of  $\lceil \log_2(d+1) \rceil + 1$  such graphs.  $\square$

Now we can use our result on bipartite graphs to get an upper bound on the irregular chromatic index of  $d$ -degenerate graphs.

**Corollary 4.4.** *If  $G$  is a connected  $d$ -degenerate graph of even size, then*

$$\chi'_{\text{irr}}(G) \leq 9(\lceil \log_2(d+1) \rceil + 1).$$

*Proof.* This follows immediately from Theorems 3.9 and 4.3.  $\square$

To get a constant upper bound for decomposable graphs in general, we combine Corollary 4.4 with Przybyło's result on graphs with large minimum degree. For this purpose, we need the following lemma.

**Lemma 4.5.** *Let  $d$  be a natural number. If  $G$  is a connected graph of even size, then  $G$  can be decomposed into two graphs  $D$  and  $H$  such that  $D$  is  $2d$ -degenerate, every connected component of  $D$  has even size, and the minimum degree of  $H$  is at least  $d$ .*

*Proof.* Starting from  $D = \emptyset$  and  $H = G$ , we remove vertices of degree at most  $2d$  from  $H$  and add them to  $D$ . Once this process stops, the graph  $D$  is  $2d$ -degenerate and  $H$

has minimum degree at least  $2d + 1$ . Every connected component  $C$  of  $D$  with odd size intersects  $H$ ; let  $v(C)$  be a vertex in the intersection. Notice that  $v(C) \neq v(C')$  for different connected components  $C$  and  $C'$  of  $D$ . We choose an almost-balanced orientation of  $H$ , i.e. an orientation where the out-degree and in-degree at every vertex differ by at most 1. For each connected component  $C$  of odd size, we choose an out-edge  $e(C)$  at  $v(C)$  in  $H$ . We remove  $e(C)$  from  $H$  and add it to  $D$ . Since every vertex in  $H$  might lose all of its in-edges but at most one out-edge, the minimum degree in  $H$  remains at least  $d$ . The edges we add to  $D$  in this step induce a 2-degenerate subgraph, so  $D$  will still be  $2d$ -degenerate. Moreover, every connected component of odd size gains an edge and possibly gets joined to other connected components of even size. In any case, all connected components of  $D$  now have even size.  $\square$

Now we are ready for the proof of our main result.

**Theorem 4.6.** *If  $G$  is a decomposable graph, then  $\chi'_{\text{irr}}(G) \leq 328$ .*

*Proof.* By Theorem 2.3 it suffices to show that  $\chi'_{\text{irr}}(G) \leq 327$  holds for connected graphs  $G$  of even size. By Lemma 4.5, we can decompose  $G$  into two graphs  $D$  and  $H$  such that  $D$  is  $(2 \cdot 10^{10})$ -degenerate, every connected component of  $D$  has even size, and the minimum degree of  $H$  is at least  $10^{10}$ . By Theorem 1.4, we have  $\chi'_{\text{irr}}(H) \leq 3$  and by Corollary 4.4 we have

$$\chi'_{\text{irr}}(D) \leq 9(\lceil \log_2(2 \cdot 10^{10} + 1) \rceil + 1) = 324.$$

Hence,  $\chi'_{\text{irr}}(G) \leq \chi'_{\text{irr}}(H) + \chi'_{\text{irr}}(D) \leq 3 + 324 = 327$ .  $\square$

## 5 Decomposing highly edge-connected bipartite graphs

Our bound on the irregular chromatic index of decomposable graphs in Theorem 4.6, depends partly on the irregular chromatic index of bipartite graphs. In particular, decreasing our bound in Theorem 3.9 from 9 down to 3 (as suggested by Conjecture 1.2) would already yield an improvement on the constant in Theorem 4.6. The restriction of Conjecture 1.2 to bipartite graphs, however, appears to be a surprisingly non-trivial problem.

Another interesting question concerning bipartite graphs and locally irregular decompositions is about whether the family of bipartite graphs with irregular chromatic index at most 2 admits an “easy” characterisation. It is legitimate to raise this question, as trees admit such a characterisation, see [2]. We also note that a similar study of bipartite graphs  $G$  satisfying  $\chi'_{\Sigma}(G) \leq 2$  was recently conducted by Thomassen, Wu, and Zhang [11], resulting in such a characterisation. So far all bipartite graphs  $G$  with irregular chromatic index 3 we know have minimum degree 1 or 2. It could be the case that minimum degree 3 already suffices to push the irregular chromatic index down to 2 for bipartite graphs. Notice that  $d$ -regular bipartite graphs  $G$  with  $d \geq 3$  do indeed satisfy  $\chi'_{\text{irr}}(G) = 2$ , as was shown in [1].

**Question 5.1.** *Does there exist a bipartite graph  $G$  with minimum degree at least 3 and  $\chi'_{\text{irr}}(G) > 2$ ?*

In this section we prove that 16-edge-connected bipartite graphs have irregular chromatic index at most 2. The main tool is the following result on factors modulo  $k$  in bipartite graphs, due to Thomassen [10]. That result was based on the proof of the weak version of Jaeger’s Circular Flow Conjecture in [9]. The quadratic bound on the edge-connectivity in [9] was improved to a linear bound in [6].

**Theorem 5.2** ([10]). *Let  $k$  be a natural number, and let  $G$  be a  $(3k - 2)$ -edge-connected bipartite graph with partition classes  $A$  and  $B$ . If  $f : V(G) \rightarrow \mathbb{Z}$  is a function satisfying*

$$\sum_{v \in A} f(v) \equiv \sum_{v \in B} f(v) \pmod{k},$$

*then  $G$  has a spanning subgraph  $H$  with  $d(v, H) \equiv f(v) \pmod{k}$  for every  $v \in V(G)$ .*

Theorem 1 in [9] assumes edge-connectivity  $3k - 3$ . But this holds only for  $k$  odd. For  $k$  even it should be  $3k - 2$ , see page 11 in [6], and below we shall apply Theorem 5.2 for  $k = 6$ .

**Theorem 5.3.** *For every 16-edge-connected bipartite graph  $G$ , we have  $\chi'_{\text{irr}}(G) \leq 2$ .*

*Proof.* Let  $A$  and  $B$  denote the partition classes of  $G$ . Since  $G$  is 16-edge-connected, we have  $|A|, |B| \geq 16$ .

Suppose first that every vertex has at most one non-neighbour in the other partition class. Let us assume that  $|A| \geq |B|$ . If  $|A| - |B| \geq 2$ , then  $G$  is already locally irregular. If  $|A| - |B| \leq 1$ , then let  $H$  be a subgraph of  $G$  consisting of two vertices in  $B$  and all edges incident with one of these vertices. Clearly  $H$  is locally irregular. Let  $H'$  denote the subgraph  $G - E(H)$ . For  $v \in A$ , we have  $d(v, H') \leq |B| - 2 \leq |A| - 2$ . For  $v \in B$  we have  $d(v, H') \geq |A| - 1$ , so  $H'$  is locally irregular and  $\chi'_{\text{irr}}(G) \leq 2$ .

Now suppose that  $G$  contains a vertex  $u$  with at least two non-neighbours in the other partition class. We may assume  $u \in A$ . We denote by  $A_0$  and  $B_0$  the subsets of  $A$  and  $B$  consisting of the vertices of even degree, and by  $A_1$  and  $B_1$ , respectively, the subsets consisting of the vertices of odd degree. For any function  $g : V(G) \rightarrow \mathbb{Z}$ , we define  $\sigma_A(g) = \sum_{v \in A} g(v)$  and  $\sigma_B(g) = \sum_{v \in B} g(v)$ . If  $G$  is  $(3k - 2)$ -edge-connected and  $\sigma_A(g) \equiv \sigma_B(g) \pmod{k}$ , then, by Theorem 5.2, there exists a subgraph  $H$  of  $G$  with  $d(v, H) \equiv g(v) \pmod{k}$  for all  $v \in V(G)$ . We apply this for  $k = 6$ . We assign to each vertex  $v \in V(G)$  an integer  $f(v)$  in the following way:

- for every  $v \in A_0$ , we set  $f(v) = 0$ ;
- for every  $v \in A_1$ , we set  $f(v) = 1$ ;
- for every  $v \in B_0$ , we set  $f(v) = 3$ ;
- for every  $v \in B_1$ , we set  $f(v) = 2$ .

If  $\sigma_A(f) \equiv \sigma_B(f) \pmod{6}$ , then we apply Theorem 5.2 to find a subgraph  $H$  of  $G$  with  $d(v, H) \equiv f(v) \pmod{6}$  for all  $v \in V(G)$ . Since the degrees of the vertices in  $A$  have different residues modulo 6 than the degrees of the vertices in  $B$ , the graph  $H$  is locally irregular. Let  $H'$  denote the subgraph  $G - E(H)$ . The vertices in  $A$  in  $H'$  have even degrees, while the vertices in  $B$  have odd degrees. Thus, also  $H'$  is locally irregular and  $\chi'_{\text{irr}}(G) \leq 2$ .

We may therefore assume  $\sigma_A(f) \not\equiv \sigma_B(f) \pmod{6}$ . First, suppose  $\sigma_A(f) \equiv \sigma_B(f) \pmod{2}$ . Then  $\sigma_A(f) \equiv \sigma_B(f) + 2$  or  $\sigma_A(f) \equiv \sigma_B(f) + 4$ . Let  $x$  and  $y$  be two different vertices in  $B$ . We define two new functions  $f_1$  and  $f_2$  by setting  $f_1(x) = f_2(x) = f(x) + 2$ ,  $f_2(y) = f(y) + 2$  and setting  $f_1(v) = f(v)$  for all  $v \in V(G) \setminus \{x\}$  and  $f_2(v) = f(v)$  for all  $v \in V(G) \setminus \{x, y\}$ . Now  $\sigma_A(f_1) = \sigma_A(f_2) = \sigma_A(f)$  and  $\sigma_B(f_2) = \sigma_B(f_1) + 2 = \sigma_B(f) + 4$ . Thus, one of the functions  $f_1$  or  $f_2$  satisfies the condition in Theorem 5.2 and the same argument as above yields a decomposition into two locally irregular subgraphs.

Finally, suppose  $\sigma_A(f) \not\equiv \sigma_B(f) \pmod{2}$ . We define a new function  $g$  by setting  $g(u) = 1 - f(u)$ , where  $u$  is the special vertex with at least two non-neighbours in  $B$ . We set  $g(v) = f(v)$  for all  $v \in A$  and all  $v \in B$  that are non-neighbours of  $u$ . If  $v \in B$  is a neighbour of  $u$  with  $d(v) - f(v) \not\equiv d(u) - g(u) \pmod{6}$ , then we also set  $g(v) = f(v)$ . In the case that  $v \in B$  is a neighbour of  $u$  with  $d(v) - f(v) \equiv d(u) - g(u) \pmod{6}$ , we set  $g(v) = f(v) + 2$ . Now we have  $\sigma_A(g) \equiv \sigma_A(f) + 1$  and  $\sigma_B(g) \equiv \sigma_B(f)$ , so  $\sigma_A(g) \equiv \sigma_B(g) \pmod{2}$ .

If  $\sigma_A(g) \equiv \sigma_B(g) \pmod{6}$ , then we use Theorem 5.2 for the function  $g$ . Let  $H$  be the subgraph of  $G$  with  $d(v, H) \equiv g(v) \pmod{6}$  for all  $v \in V(G)$ . Notice that  $g(v) \in \{0, 1\}$  for  $v \in A$  and  $g(v) \in \{2, 3, 4, 5\}$  for  $v \in B$ , so  $H$  is locally irregular. Let  $H'$  denote the subgraph  $G - E(H)$ . Notice that in  $H'$  all vertices in  $B$  have odd degree, while  $u$  is the only vertex in  $A$  of odd degree. Notice that for every neighbour  $v$  of  $u$  we have  $d(v, G) - g(v) \not\equiv d(u, G) - g(u) \pmod{6}$  by definition of  $g$ , so  $u$  is not adjacent to a vertex of the same degree in  $H'$ . Thus, also  $H'$  is locally irregular.

If  $\sigma_A(g) \not\equiv \sigma_B(g) \pmod{6}$ , then we define functions  $g_1$  and  $g_2$  as before by adding 2 to the values of one or two vertices in  $B$ , and use Theorem 5.2 for the function  $g_1$  or  $g_2$ . This time we only choose vertices that are non-neighbours of  $u$ , to make sure the value at a vertex never increases by more than 2 compared to its original  $f$ -value. As before, this yields a decomposition of  $G$  into two locally irregular subgraphs.  $\square$

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